Virtual Element Method for the Nonlinear Convection-Diffusion-Reaction Equation

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The Model Problem

We consider the nonlinear convection-diffusion-reaction equation

$$\begin{cases} \sigma u - \nabla\cdot(K\nabla u) + \mathbf{b}\cdot\nabla u + g(u) = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $K \in L^\infty(\Omega)$, $\mathbf{b} \in W^{1,\infty}(\Omega)^2$, $\sigma \in \mathbb{R}$ and $f \in L^2(\Omega)$ with $\sigma > 0$ and $(\nabla\cdot\mathbf{b})(x) = 0$, $K(x) \geq K_0 > 0$, a.e. in Ω. We also assume that the nonlinearity $g \in C^1(\mathbb{R})$ with $g(0) = 0$ and $g'(x) \geq g_0 \geq 0$ for $x \geq 0$.

Let $V_h = \{v_h \in H_0^1(\Omega) : v_h \in H^2(E), \forall E \in \mathcal{T}_h\}$. The weak form of the SUPG technique is as follows,

$$A_{SUPG}(u_h, v_h) = F_{SUPG}(v_h) \quad \forall u_h, v_h \in V_h$$

where

$$\begin{aligned} A_{SUPG}(u_h, v_h) =& A_F(u_h, v_h) + S_h(u_h, v_h) \\ A_F(u_h, v_h) =& \int_\Omega \sigma\, u_h\, v_h + \int_\Omega K\,\nabla u_h\cdot\nabla v_h + \int_\Omega \mathbf{b}\cdot\nabla u_h\, v_h + \int_\Omega g(u_h)\, v_h \\ S_h(u_h, v_h) =& \sum_{E\in\mathcal{T}_h} \tau_E\left(\sigma u_h - \nabla\cdot(K\nabla u_h) + g(u_h), \mathbf{b}\cdot\nabla v_h\right) + \sum_{E\in\mathcal{T}_h} \tau_E\left(\mathbf{b}\cdot\nabla u_h, \mathbf{b}\cdot\nabla v_h\right). \end{aligned}$$

and $F_{SUPG}(v_h) = (f, v_h) + \sum_{E\in\mathcal{T}_h} \tau_E(f, \mathbf{b}\cdot\nabla v_h)$

The stability parameter τ_E is defined as $\tau_E = \frac{h_E}{2\,\mathbf{b}_E}\min\{Pe_E, 1\}$ where Pe_E [2] is the mesh Peclet number and $b_E = \sup_{x\in E}\|\mathbf{b}(x)\|_{\mathbb{R}^2}$.

VEM Discretisation

Let $\Pi_p^\nabla : H^1(E) \to Q_p(E)$ be the projection operator such that for every $u \in H^1(E)$,

$$\left(\nabla q, \nabla\left(\Pi_p^\nabla u_h - u_h\right)\right) = 0, \quad \forall q \in Q_p(E)$$

and $\Pi_k^0 : H^1(E) \to Q_p(E)$ be the projection operator such that for every $u \in H^1(E)$,

$$\left(q, u_h - \Pi_k^0 u - h,\right)_{0,E} = 0, \quad \forall q \in Q_p(E).$$

The local virtual element space of order p is defined as

$$\begin{aligned} V_p(E) =& \{u_h \in H^1(E) : \Delta u_h \in Q_p(E),\, u_h|_e \in Q_p(e)\ \forall e \subseteq E, \\ & (u_h, q) = \left(\Pi_k^\nabla u_h, q\right) \forall q \in Q_p(E)/Q_{p-2}(E)\} \end{aligned}$$

Now we can define the global VEM space V_h^p as

$$V_h^p = \left\{u_h \in H_0^1(\Omega) : u_h|_E \in V_p(E), \forall E \in \mathcal{T}_h\right\}$$

where $Q_p(E)/Q_{p-2}(E)$ is the subspace of $Q_p(E)$ containing polynomials orthogonal to $Q_{p-2}(E)$.

The discretization of the VEM-SUPG is as follows,

$$A_{VSG}(u_h, v_h) = F_{VSG}(v_h), \quad \forall u_h, v_h \in V_h^p$$

$$\text{where} \quad \begin{aligned} A_{VSG}(u_h, v_h) =& A1(u_h, v_h) + A2(u_h, v_h) + A3(u_h, v_h) + B1(u_h, v_h) + B2(u_h, v_h) + \\ & B3(u_h, v_h) + S1(u_h, v_h) + S2(u_h, v_h) \\ F_{VSG}(v_h) =& \left(f, \Pi_p^0 v_h\right) + \sum_{E\in\mathcal{T}_h}\tau_E\left(f, \mathbf{b}\cdot\Pi_{p-1}^0\nabla v_h\right). \end{aligned}$$

and

$$\begin{aligned} A1(u_h, v_h) &= \left(\sigma\,\Pi_{p-1}^0 u_h, \Pi_{p-1}^0 v_h\right) + \left(K\Pi_{p-1}^0\nabla u_h, \Pi_{p-1}^0\nabla v_h\right) \\ A2(u_h, v_h) &= \frac{1}{2}\left[\left(\mathbf{b}\cdot\Pi_{p-1}^0\nabla u_h, \Pi_p^0 v_h\right) - \left(\Pi_p^0 u_h, \mathbf{b}\cdot\Pi_{p-1}^0\nabla v_h\right)\right] \\ A3(u_h, v_h) &= \left(g(\Pi_p^0 u_h), \Pi_p^0 v_h\right) \\ B1(u_h, v_h) &= \sum_{E\in\mathcal{T}_h}\tau_E\left(\sigma\,\Pi_p^0 u_h - \nabla\cdot K\Pi_{p-1}^0\nabla u_h, \mathbf{b}\cdot\Pi_{p-1}^0\nabla v_h\right) \\ B2(u_h, v_h) &= \sum_{E\in\mathcal{T}_h}\tau_E\left(\mathbf{b}\cdot\Pi_{p-1}^0\nabla u_h, \mathbf{b}\cdot\Pi_{p-1}^0\nabla v_h\right) \\ B3(u_h, v_h) &= \sum_{E\in\mathcal{T}_h}\tau_E\left(g(\Pi_p^0 u_h), \mathbf{b}\cdot\Pi_{p-1}^0\nabla v_h\right) \\ S1(u_h, v_h) &= \left(K_E + \tau_E\, b_E^2\right) S^E\left((I - \Pi_p^\nabla)u_h, (I - \Pi_p^\nabla)v_h\right) \\ S2(u_h, v_h) &= (\sigma + g_0)\, S^E\left((I - \Pi_p^0)u_h, (I - \Pi_p^0)v_h\right) \end{aligned}$$

where, $K_E := \sup_{x\in E} K(x)$, $K_E^\vee := \inf_{x\in E} K(x)$ and S^E is the symmetric bilinear form defined in [2].

Results

For analysing the scheme, we consider the following appropriate norm,

$$|||v||| := \left(\sum_{E\in\mathcal{T}_h}\left(||\sqrt{K}\nabla v||_{L^2(E)}^2 + (\alpha + g0)||v||_{L^2(E)}^2 + \tau_E||\mathbf{b}\cdot\nabla v||_{L^2(E)}^2\right)\right)^{1/2}.$$

Coercivity : Assume the condition $0 \leqslant \tau_E \leqslant \frac{1}{4}\min\left\{\frac{h_E^2}{p_E^4\mu_{inv}^2 K_E}; \frac{1}{\sigma}; \frac{\sigma+g_0}{L_g^2}\right\}$, where L_g is the Lipschitz constant of g. Then, $A_{VSG}(v_h, v_h) \geqslant \frac{1}{4}|||v_h|||^2 \quad \forall v_h \in V_h^p$.

Using Brouwer's fixed point theorem along with the coercivity, we have the following,

Stability : Let assumptions on τ_E be satisfied. Then, the VEM-SUPG scheme admits a solution $u_h \in V_h^p$ satisfying

$$|||u_h||| \leqslant C|||f|||_*,$$

with the dual norm $|||f|||_* := \sup_{v_h\in V_h^p\setminus\mathbf{0}} F_{VSG}(v_h)/|||v_h|||$.

Error estimate : Let $u \in \mathrm{H}_0^1(\Omega)$ be the solution with $u \in \mathrm{H}^s(E), s > 1$, for all $E \in \mathcal{T}_h$. Suppose that $I_{h,p}u \in V_h^p$. Then, for the VEM-SUPG scheme it holds that

$$\begin{aligned} |||u - u_h|||^2 \leqslant& C\sum_{E\in\mathcal{T}_h}\frac{h_E^{2(s-1)}}{p^{2(s-1)}}||u||_{\mathrm{H}^s(E)}^2 \\ & \left(K_E + \frac{(\sigma + g_0)h_E^2}{p^2} + \tau_E\mathbf{b}_E + \min\left\{\left(\max_{E\in\mathcal{T}_h}\left(\frac{\mathbf{b}_E^2}{\sigma K_E^\vee}\right)\right)^{\frac{1}{2}}\frac{1}{\tau_E}; \frac{\mathbf{b}_E^2}{K_E^\vee}\right\}\frac{h_E^2}{p^2}\right). \end{aligned}$$

Numerical Experiment

In the VEM discretisation, we consider also an alternate option for the term $A2(u_h, v_h)$, which we denote $\widetilde{A2}(u_h, v_h) := \left(\mathbf{b}\cdot\Pi_{p-1}^0\nabla u_h,\ \Pi_p^0 v_h\right)$. Outcomes for both the options are shown and they are almost similar. Since the equation is nonlinear, Newton's method was employed to solve the arising nonlinear system of equations. The stopping criteria for Newton's loop was set as 10^{-6}.

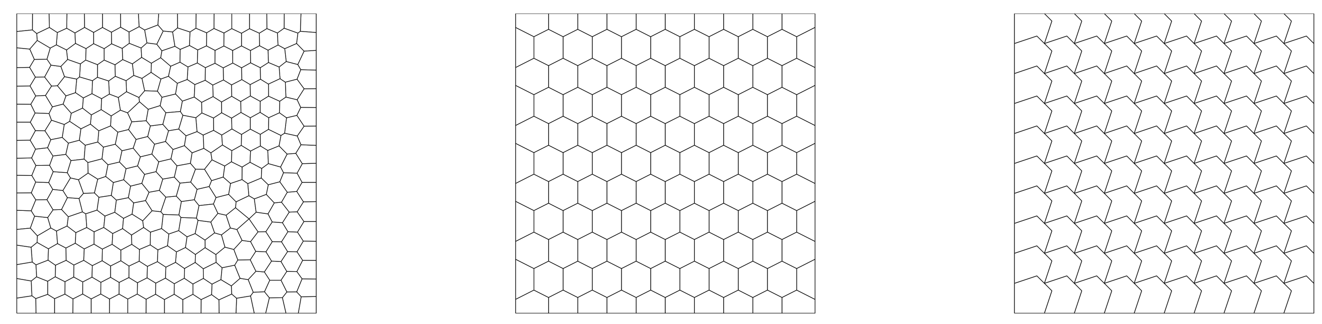

Figure 1: Polygonal meshes : Voronoi($left$, hexagon($center$) and nonconvex($right$) mesh.

The domain is discretised using a family of the polygonal meshes, with mesh diameter in the range $10^{-1} - 10^{-2}$. A sample mesh for each family is shown in figure 2.

Problem 1

We have $\Omega = (0,1)^2$ with $\sigma = 12$, $\mathbf{b}(x) = (2,3)$, and $g(u) = u^3$. The source term f is determined such that the solution u is given by

$$\begin{aligned} u(\mathbf{x}, t) =& 16x_1(1 - x_1)x_2(1 - x_2)\times \\ & \left[0.5 + \pi^{-1}\text{arctan}\left(200\left(0.25^2 - (x_1 - 0.5)^2 - (x_2 - 0.5)^2\right)\right)\right]. \end{aligned} \tag{2}$$

In this example, we conduct numerical test for three different small diffusion coefficients, namely, $\epsilon = 10^{-3}$, $\epsilon = 10^{-6}$, and $\epsilon = 10^{-9}$. We use Dirichlet boundary values prescribed by the exact solution. The solution admits circular interior layer. The convergence rates are presented for both the options $A2$ and $\widetilde{A2}$. It is observed that similar results are obtained for both these options, on the considered meshes. The expected rate of convergence achieved for the VEM orders $p = 1$ and $p = 2$, for $\epsilon = 10^{-6}$ on Voronoi mesh is presented. We mention (not shown here) that similar rates were obtained for the other epsilon($\epsilon = 10^{-3}, 10^{-6}$) as well.

	Voronoi mesh, p = 1, $\epsilon = 10^{-6}$							
h	$e_{h,0}$	$e_{h,1}$	$r_{h,0}$	$r_{h,1}$	$\tilde{e}_{h,0}$	$\tilde{e}_{h,1}$	$\tilde{r}_{h,0}$	$\tilde{r}_{h,1}$
0.197119	6.700985e-02	3.100322	-	-	6.718384e-02	3.100649	-	-
0.105796	3.798898e-02	2.595629	0.82	0.26	3.800486e-02	2.595668	0.82	0.26
0.051281	1.590650e-02	1.854286	1.25	0.48	1.594259e-02	1.857973	1.25	0.48
0.026065	5.577630e-03	1.123444	1.51	0.72	5.509496e-03	1.119906	1.53	0.73
0.012514	1.291895e-03	5.288503e-01	2.11	1.09	1.246919e-03	5.252897e-01	2.14	1.09

	Voronoi mesh, p = 2, $\epsilon = 10^{-6}$							
h	$e_{h,0}$	$e_{h,1}$	$r_{h,0}$	$r_{h,1}$	$\tilde{e}_{h,0}$	$\tilde{e}_{h,1}$	$\tilde{r}_{h,0}$	$\tilde{r}_{h,1}$
0.197119	3.814583e-02	2.623398	-	-	3.827998e-02	2.626769	-	-
0.105796	1.639883e-02	1.829960	1.22	0.52	1.640535e-02	1.831642	1.22	0.52
0.051281	4.716297e-03	9.457611e-01	1.79	0.95	4.734615e-03	9.492477e-01	1.79	0.95
0.026065	1.093927e-03	3.766024e-01	2.11	1.33	1.092445e-03	3.795338e-01	2.11	1.32
0.012514	9.931297e-05	8.432417e-02	3.46	2.16	9.522849e-05	8.346176e-02	3.52	2.18

References

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