# Classical Optic Radial-Angular Entanglement

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**Abstract** We study the classical optic entanglement between the radial and angular degrees of freedom in Laguerre-Gaussian mode superpositions with regard to symmetric first-order optical systems. The role of Gouy phase picked by a Laguerre-Gaussian mode on free propagation in regard to the radial-angular entanglement in the mode superpositions is brought out. We obtain a witness of radial-angular entanglement in two mode Laguerre-Gaussian superpositions which is demonstrated to be a robust free space signaler in the presence of atmospheric turbulence.

## Introduction

Paraxial light fields are solutions of the paraxial wave equation and retain their paraxiality on passage through symmetric first-order optical systems. Examples of such light fields are the Laguerre-Gaussian (LG) modes.

$\Psi_{jm}(r,\theta) = \psi_{jm}(r;q) \times \Theta_m(\theta)$, where

$$\psi_{jm}(r;q) = \sqrt{\frac{2}{\pi w^2}}\left(\frac{(j-|m|)!}{(j+|m|)!}\right)^{\frac{1}{2}} \times \left(\frac{\sqrt{2}\,r}{w}\right)^{2|m|} L_{j-|m|}^{2|m|}\left(\frac{2\,r^2}{w^2}\right)\exp\left[\frac{i\,\pi r^\wedge 2}{\lambda q}\right],$$

with $\frac{1}{q} = \frac{1}{R} + \frac{i\,\lambda}{\pi w^\wedge 2}$ and $\Theta_m(\theta) = \exp[i2m\theta]$

Here $L_{j-|m|}^{2|m|}(.)$ is the Laguerre polynomial with radial and azimuthal indices $j$ and $m$, $w$ is the width of the beam waist, $\lambda$ is the wavelength, and $q$ is the complex field parameter. For an LG mode propagating from the waist plane, we have

$w^2 \equiv w^2(d) = w^2(0)\left[1+\left(\frac{d}{d_r}\right)^2\right]$, and $R \equiv R(d) = d\left[1+\left(\frac{d_r}{d}\right)^2\right]$,

where $d$ is the distance of propagation from the waist plane, $w(0)$ is the waist plane width, and $d_r = \frac{\pi w(0)^2}{\lambda}$.

## Unitary Transformations

Free propagation unitary transformation:

$U_f(d) = \exp\left[-i\frac{d\lambda}{4\pi}\nabla_\perp^2\right]$, where $\nabla_\perp^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

Thin lens unitary transformation:

$U_l(f) = \exp\left[\frac{-i\pi}{\lambda f}r^2\right]$

Action of free propagation unitary on an LG mode:

$U_f(d)\psi_{jm}(r;q)\Theta_\mathrm{m}(\theta) = e^{-i\phi_j(q;d)}\psi_{jm}(r;q')\Theta_m(\theta)$, where $q' = q + d$, and

$\phi_j(q;d) = -(2j+1)\tan^{-1}\left(\frac{\frac{\lambda}{\pi w^2}}{\frac{1}{d}+\frac{1}{R}}\right)$

Action of thin lens unitary on an LG mode:

$U_l(f)\psi_{jm}(r;q)\Theta_\mathrm{m}(\theta) = \psi_{jm}(r;q')\Theta_m(\theta)$, where $\frac{1}{q'} = \frac{1}{q} - \frac{1}{d}$

Any symmetric first-order optical system is realized as a sequence of free propagations and thin lenses.

## Classical Optic Radial-angular Entanglement

A paraxial field $\Psi(r,\theta)$, which is a finite superposition of LG modes, on Schmidt decomposition gives:

$$\Psi(r,\theta) = \sum_{j,m} c_{jm}\,\Psi_{jm}(r,\theta) = \sum_{j,m} d_{jm}\,\psi_j(r;q)\Theta_\mathrm{m}(\theta)$$

We define $(j,m)^{\text{th}}$ matrix entry of $D$ as $(D)_{j,m} = d_{jm}$ and $\Lambda = D^\dagger D$. The radial-angular entanglement $E(\Psi)$ of $\Psi(r,\theta)$ is defined through the eigenvalues of $\Lambda$ as

$E(\Psi) = -\sum_i \lambda_i \log\lambda_i$.

## Observations

- For any mode superposition $\Psi(r,\theta)$, the radial-angular entanglement is invariant under transverse plane rotation.
- For any mode superposition $\Psi(r,\theta)$ with constituent modes of distinct $j$ and $m$ indices, the radial-angular entanglement in $\Psi(r,\theta)$ is invariant on passage through any symmetric first-order optical system.
- For any mode superposition $\Psi(r,\theta)$, with constituent modes having a fixed $j$ index and varying $m$ index, the radial-angular entanglement in $\Psi(r,\theta)$ is invariant on passage through a symmetric first-order optical system.
- Free propagation can in general create, preserve, or destroy radial-angular entanglement in a mode superposition $\Psi(r,\theta)$.

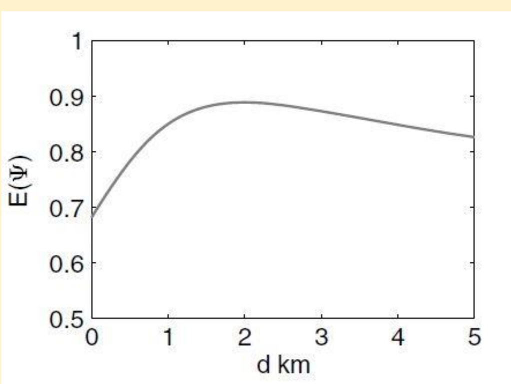


**Fig. 1.** $E(\Psi)$ against $d$ for the three-mode superposition $\Psi(r,\theta) = \frac{1}{\sqrt{3}}\Psi_{00}(r,\theta) + \frac{1}{\sqrt{3}}\Psi_{11}(r,\theta) + \frac{-\mathrm{i}}{\sqrt{3}}\Psi_{21}(r,\theta)$.

## Witness of Radial-angular Entanglement

We define a measurable quantity, $S$ which serves as a witness of radial-angular entanglement, which is also a possible free space signaler.

$$S = \left\langle \frac{r^2}{w^2(d)}\frac{-i\partial}{\partial\theta}\right\rangle - \left\langle\frac{-i\partial}{\partial\theta}\right\rangle\left\langle\frac{r^2}{w^2(d)}\right\rangle$$

For a two mode superposition, $\Psi(r,\theta) = c_{j_1m_1}\Psi_{j_1m_1}(r,\theta) + c_{j_2m_2}\Psi_{j_2m_2}(r,\theta)$

$$S_{j_2m_2}^{j_1m_1} = 2\left|c_{j_1m_1}\right|^2\left|c_{j_2m_2}\right|^2(\mathrm{j}_1-\mathrm{j}_2)(\mathrm{m}_1-\mathrm{m}_2).$$

## Propagation through Atmospheric Turbulence

While both $S$ and $E(\Psi)$ remain invariant on free propagation for a two mode superposition, they need not be invariant as the superposition propagates through atmospheric turbulence. The phase spectrum of a two dimensional random phase can be written in terms of the Kolmogorov power spectral density as:

$\Phi_\theta(K) = 2\pi\left(\frac{2\pi}{\lambda}\right)^2\delta_d\Phi_n(K)$ where $\delta_d$ is length of the interval, with

$$\Phi_n(K) = 0.033\,C_n^2\,K^{-11/3}$$

## Example:

Consider the two mode LG superposition with mode indices $j_1, m_1 = 0, 0$ and $j_2, m_2 = \frac{3n}{2}, \frac{n}{2}$, i.e.,

$$\Psi^{1n}(r,\theta) = \sqrt{(n-1)/n}\,\Psi_{00}(r,\theta) + \sqrt{1/n}\,\Psi_{\frac{3n}{2}\frac{n}{2}}(r,\theta) \qquad \textbf{(1)}$$

The witness of radial-angular entanglement for such a mode superposition is given by $S_{\frac{3n}{2}\frac{n}{2}}^{00} = \frac{3(n-1)}{2}$.

## Simulation Results

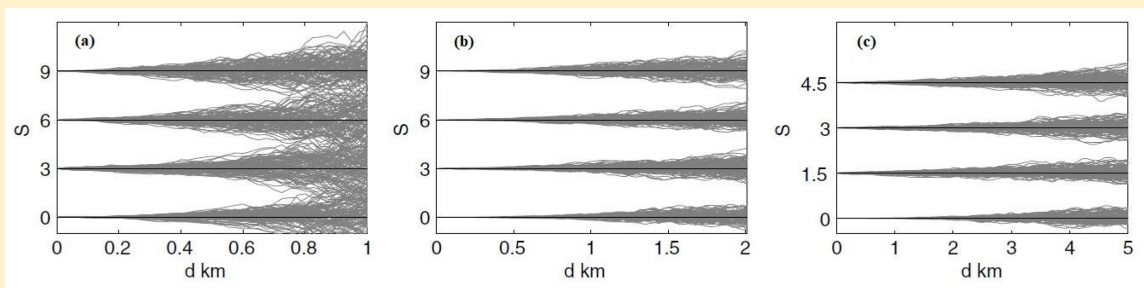


**Fig. 2.** (a), (b) Witness of radial–angular entanglement $S$ against distance of propagation $d$ for 100 samples for $n$ = 1,3, 5, and 7 and $C_n^2 \approx 10^{-12}$ $\mathrm{m}^{-2/3}$ and $10^{-13}$ $\mathrm{m}^{-2/3}$, respectively, for $\Psi^{1n}$ given in Eq. (1), for $d$ up to 1 km and 2 km, respectively; (c) same, but for $n$ = 1,2, 3, and 4 and $C_n^2 \approx 10^{-14}$ $\mathrm{m}^{-2/3}$, and for $d$ up to 5 km for $\Psi^{1n}(r,\theta)$. The dark line in each of these frames shows $S$ for the respective $\Psi^{1n}(r,\theta)$ in the absence of atmospheric turbulence.

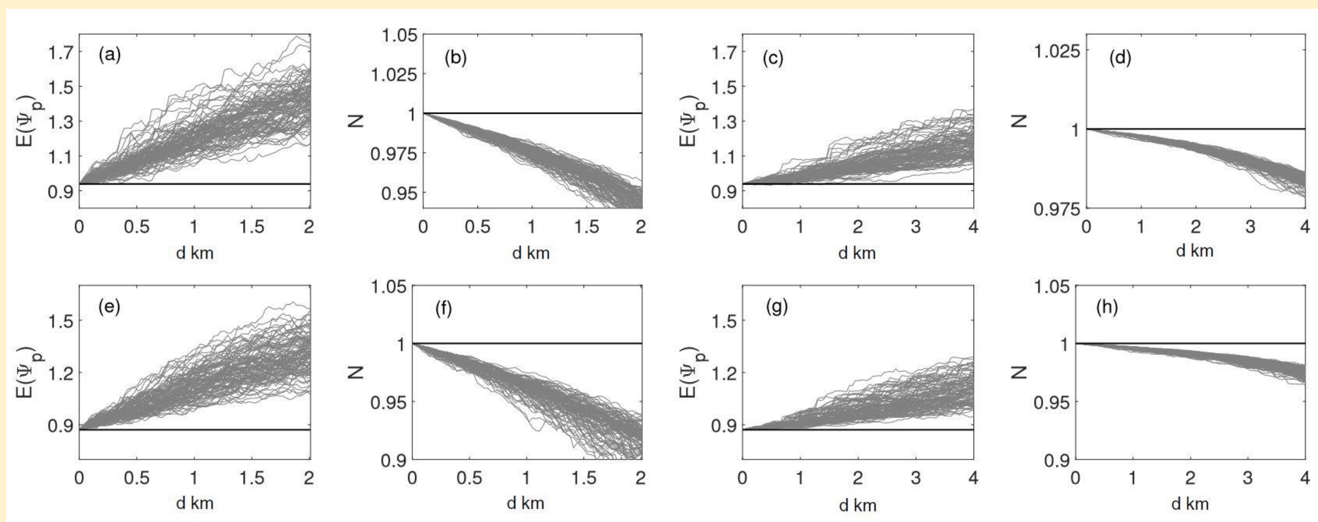


**Fig. 3.** (a), (c) $E(\Psi_p)$ against distance of propagation $d$ for $\Psi^{1n}(r,\theta)$ with $n$ = 2 [Eq. (1)] and $C_n^2 \approx 10^{-13}$ $\mathrm{m}^{-2/3}$ and $10^{-14}$ $\mathrm{m}^{-2/3}$, respectively, for 100 samples, for $d$ up to 2 km and 4 km; (b), (d) $N$ against distance of propagation $d$ for the samples in (a) and (c), and corresponding strengths of turbulence; (e), (g) [and (f ), (h)] same, but for $\Psi^{1n}(r,\theta)$ with $n$ = 3 for the respective strengths of turbulence. In all the frames, the dark line shows either $E(\Psi)$ or $N(=1)$ in the absence of atmospheric turbulence.

## Conclusion

To conclude, we have studied radial-angular entanglement in LG mode superpositions with respect to symmetric first-order optical systems. The role of the Gouy phase in this regard has been brought out. We have seen examples of LG mode superpositions for which a symmetric first-order optical system preserves the radial-angular entanglement. The examples studied suggest that atmospheric turbulence can alter the radial-angular entanglement in a mode superposition on propagation through atmospheric turbulence. We have illustrated through examples how the defined witness of radial-angular entanglement can be effective as a free space signaler. This suggests that classical optic entanglement can indeed be of practical value, particularly in the context of free space optical communication.

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